Mixed Balancing Truncation for Better Impulse Response

Kwang Young Jeong*
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Two methods of balancing reduction are presented. The first is to truncate the states corresponding to smaller Hankel singular values of the reciprocal balanced system. The second is to choose one of some balancing methods, that has the least impulse response error by reducing one order at a time. The relation between the grammians of the balanced system and those of the reciprocal system is derived.

Key Words : Balancing Reduction, Balanced Realization, Hankel Singular Value, Impulse Response

1. Introduction

Recent control literature shows that an important role is played by the balanced realization truncation order reduction techniques in model and controller reduction. In this realization, controllability and observability grammians are equal and diagonal. The diagonal elements are called Hankel singular values. Moore (1981) obtained a reduced-order model by retaining states corresponding to larger Hankel singular values. Kabamba (1985) and Davidson (1986) suggested alternate criteria in which the states that contribute significantly to the impulse response norm of the original system are retained in the reduced-order model.

Reciprocal transformations suggested by Fernando and Nicholson (1983) and Sreeram and Agathoklis (1989) were developed to give better approximation in the low frequency range. In their examples, these methods gave smaller impulse response error as well as smaller steady state error than Kabamba's method. Generalizations of the reciprocal transformations were shown by Muscato and Nunnari (1994) and Clapperton et al. (1994).

In this paper, two new reduction methods are presented, which are motivated by the following two observations. First, depending on the problem itself, Moore's truncation criteria may give better impulse response than Davidson's criteria. Secondly, Sreeram and Agathoklis adopted Davidson's criteria for truncation in the reciprocal balanced system.

The first proposed method is based on forming the reciprocal transfer function, computing the reduced model according to Hankel singular values and reciprocating it back to get the required reduced order model. This idea of reducing a reciprocal transfer function and reciprocating back the reduced order model, is common in frequency domain techniques of model reduction. (Sreeram and Agathoklis, 1989)

The second proposed method derives reduced-order models according to Moore's, Davidson's, Sreeram and Agathoklis's and the first proposed method by truncating one state. A reduced-order model that has the least impulse response error is selected at this step. Restarting from this model, the next reduced-order model is derived by reducing one order at a time until a desired order reduced-order model is found.

The major difficulty of the second proposed method is a large amount of computing time. In order to apply it, it is necessary to find two balanced systems at each step; one for the current system, the other for its reciprocal system. The relation between the grammians of the balanced

* Assistant Professor, Dept. of Mechanical Engineering, Kongju National University, Kongju, Chungnam, Korea

system and those of the reciprocal system will be derived to reduce the computing time of the second method.

2. Balancing Reduction Methods

In this section, the three reduction methods based on the balancing coordinates for a linear time-invariant system are briefly described.

Let us suppose a stable linear time-invariant system (A, B, C), which is described as:

$$\dot{x}=Ax+Bu$$
$$y=Cx \tag{1}$$

where $x\in R^n$, $u\in R^r$ and $y\in R^s$. The system is assumed to be controllable and observable over $[0, \infty)$, which means that the controllability grammian

$$W_c=\int_0^\infty e^{At}BB^Te^{A^Tt}\mathrm{dt} \tag{2}$$

and the observability grammian

$$W_0=\int_0^\infty e^{A^Tt}C^TCe^{At}\mathrm{dt} \tag{3}$$

are both nonsingular. It is well known that these grammians satisfy the following Lyapunov equations.

$$AW_c+W_cA^T+BB^T=0 \tag{4}$$
$$A^TW_0+W_0A+C^TC=0 \tag{5}$$

The given system is transformed into a balanced system, where controllability and observability grammians are equal and diagonal (Moore, 1981).

$$\dot{\hat{x}}=\hat{A}\hat{x}+\hat{B}u$$
$$y=\hat{C}\hat{x} \tag{6}$$

where

$$\hat{A}=T^{-1}AT, \hat{B}=T^{-1}B, \ \hat{C}=CT \tag{7}$$

and

$$\hat{W}_c=\hat{W}_o=\mathrm{diag}$$
$$(\sigma_1^2, \sigma_2^2, \cdots\sigma_r^2, \sigma_{r+1}^2\cdots, \sigma_n^2)$$
$$(\sigma_1^2\geq\sigma_2^2\geq\cdots\ \sigma_r^2\gg\sigma_{r+1}^2\cdots\ \geq\sigma_n^2>0) \tag{8}$$

The states in the balanced coordinates are equally controllable and observable. The quantities σ_i^2, known as Hankel singular values, are invariants and reflect the input/output importance of the states. The reduced-order model is obtained by truncating the states corresponding to smaller Hankel singular values, i.e.,

$$\hat{A}_r=[\ I_r\ 0\ ]\hat{A}\begin{bmatrix} I_r \\ 0 \end{bmatrix}$$
$$\hat{B}_r=[\ I_r\ 0\ ]\hat{B} \text{ and } \hat{C}_r=\hat{C}\begin{bmatrix} I_r \\ 0 \end{bmatrix} \tag{9}$$

where I_r is an identity matrix with r columns and r rows.

Kabamba (1985) and Davidson (1986) suggested different truncation criteria based on the system impulse response. Consider the system impulse response norm of Eq. (1)

$$\| g \|_2^2=\int_0^\infty g^Tg\mathrm{dt} \tag{10}$$

where $g(t)$ is the unit impulse response, i.e.,

$$g=Ce^{At}B \tag{11}$$

This norm can be expressed in terms of the grammians.

$$\| g \|_2^2=\mathrm{tr}\int_0^\infty gg^T\mathrm{dt}=\mathrm{tr}(C^TCW_c)$$
$$=\mathrm{tr}(BB^TW_0) \tag{12}$$

By representing the norm in the balanced coordinates

$$\| g \|_2^2=\mathrm{tr}(\hat{C}^T\hat{C}\hat{W}_c)=\mathrm{tr}(\hat{B}\hat{B}^T\hat{W}_0)$$
$$=\sum_{i=1}^n c_i^2\sigma_i^2=\sum_{i=1}^n b_i^2\sigma_i^2=\sum_{i=1}^n d_i \tag{13}$$

where c_i and b_i are called balanced gains, the square of which are ith diagonal components of $\hat{C}^T\hat{C}$ and $\hat{B}\hat{B}^T$, respectively. Since the contribution of the states to the impulse response depends on the quantity d_i rather than σ_i^2, states corresponding to larger d_i are retained in the reduced model. Therefore, if the truncated parts in this method correspond to smaller Hankel singular values, this method generates the same reduced-order model as Moore's.

Sreeram and Agathoklis (1989) proposed another method using the reciprocal transformation. First, the reciprocal system $(\bar{A}, \bar{B}, \bar{C})$ is found using the following transformation with an assumption that A is non-defective:

$$\bar{A}=A^{-1}, \bar{B}=A^{-1}B, \bar{C}=-C \tag{14}$$

For an SISO system, the transfer function of this reciprocal system is the same as that of the original system with coefficients reversed. Secondly, the balanced system $(\bar{A}, \bar{B}, \bar{C})$ of this reciprocal system is derived. Thirdly, the reduced-order model $(\bar{A}_r, \bar{B}_r, \bar{C}_r)$ of this reciprocal system is obtained by truncating states with smaller d_i. Finally, the reduced-order model (A_r, B_r, C_r) of the original system is found by transforming back to the original coordinates; i.e.,

$$A_r = \bar{A}_r^{-1},\ B_r = \bar{A}_r^{-1}\bar{B}_r \text{ and } C_r = -\bar{C}_r \tag{15}$$

3. Reciprocal Balancing Truncation

The impulse response error (Hyland and Bernstein, 1985) is defined as

$$e = \frac{\| g - g_r \|_2}{\| g \|_2} \tag{16}$$

where g and g_r are the impulse responses of the original model (A, B, C) and reduced-order model (A_r, B_r, C_r), respectively. The square of the numerator of Eq. (16) is expanded as follows :

$$\begin{aligned} \| g - g_r \|_2^2 &= \| g \|_2^2 + \| g_r \|_2^2 - 2\mathrm{tr}\int_0^\infty g g_r^T dt \\ &= \mathrm{tr}(C^T C W_c) + \mathrm{tr}(C_r^T C_r W_{rc}) \\ &\quad -2\mathrm{tr}(C_r V C) \end{aligned} \tag{17}$$

where W_{rc} is the controllability grammian of the reduced-order model and V is the solution of the following algebraic Lyapunov equation.

$$A_r V + V A^T + B_r B^T = 0 \tag{18}$$

If the error measure is $\| g \|_2 - \| g_r \|_2$, BTG (Balancing Truncation considering Balanced Gains of Kabamba, 1985) always gives smaller error than BT (Balancing Truncation of Moore, 1981) as explained in the previous section. For the usually used error given in Eq. (16), however, any explicit decision can not be made. Experience shows that it depends on the problem itself whether BTG is better than BT or not. An example that BTG is better than BT is treated by Kabamba (1985) while a counterexample is found by Sreeram and Agathoklis (1989).

Similarly, since truncation of RBTG (Reciprocal Balancing Truncation considering Balanced Gains of Sreeram and Agathoklis, 1989) is based on balanced gains multiplied by Hankel singular values, another truncation method based only on Hankel singular values of the reciprocal system is proposed. Hereafter, this method is called RBT (Reciprocal Balancing Truncation) whose algorithm is as follows.

Algorithm

(i) Obtain the reciprocal system $(A^{-1}, A^{-1}B, -C)$ for the given system (A, B, C).

(ii) Find the balanced system $(\hat{A}, \hat{B}, \hat{C})$ of $(A^{-1}, A^{-1}B, -C)$.

(iii) Obtain the reduced-order model $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ by truncating the parts corresponding to smaller Hankel singular values.

(iv) Obtain the reciprocal model (A_r, B_r, C_r) as $(\hat{A}_r^{-1}, \hat{A}_r^{-1}\hat{B}_r, -\hat{C}_r)$.

The only difference between RBT and RBTG is the truncation criteria.

4. Mixed Balancing Truncation and Grammian Calculation

4.1 Mixed balancing truncation

The main idea of MBT (Mixed Balancing Truncation) is to truncate one state at a time by choosing one of the four methods, i.e., BT, BTG, RBTG and RBT, until a desired order reduced model is found. Suppose that we want to derive an r-th order model from nth order system. First, $(n-1)$th order models can be found using the four methods. At this step, this method selects the one that has the least impulse response error. From the next step, the same procedure is applied until the r-th order model is obtained.

Since our purpose is to get a reduced model that has the least impulse response error, MBT chooses a method that has the least impulse response error at each truncation. Although this choice may give good results, this does not guarantee that the final reduced model has less impulse response error than applying a single method. Numerical examples show that the choice based on the impulse response error gives competitive results and that the other choice may give better results.

4.2 Grammian calculation

The major difficulty of MBT is the amount of computing time required to calculate reduced-order models at each order. Since we use four methods, we need to compute the balancing transformation at each order because even though a system is balanced, the reciprocal system is not usually balanced. As Laub (1980) pointed out, the majority of the computing time for balancing transformation is to derive the controllability and observability grammians. Thus, if the relation of the grammians of one system and its reciprocal system is known, the computing time of the mixed method is greatly reduced.

For a given system (A, B, C), the reciprocal system $(\bar{A}, \bar{B}, \bar{C})$ is obtained from the following transformations:

$$\bar{A}=A^{-1},\ \bar{B}=A^{-1}B,\ \bar{C}=-C \tag{19}$$

The controllability and observability grammians of the given system satisfy Eqs. (4) and (5) and those of the reciprocal system satisfy the following equations.

$$\bar{A}\bar{W}_c+\bar{W}_w\bar{A}^T+\bar{B}\bar{B}^T=0 \tag{20}$$

$$\bar{A}^T\bar{W}_0+\bar{W}_0{}^T\bar{A}+\bar{C}^T\bar{C}=0 \tag{21}$$

The following relation between W_c and $\bar{W}_c$ is derived from Eqs. (4), (19) and (20).

$$\bar{W}_c=W_c \tag{22}$$

By applying Eq. (19) into Eq. (21), we get

$$A^{-T}\bar{W}_o+\bar{W}_oA^{-1}+C^TC=0 \tag{23}$$

If we replace $\bar{W}_o$ by A^TW_oA in Eq. (23), we get Eq. (5). That is to say, the relation between $\bar{W}_o$ and W_o is

$$\bar{W}_o=A^TW_oA \tag{24}$$

The uniqueness of $\bar{W}_o$ in Eq. (21) is guaranteed if Eq. (5) has a unique solution. That is to say, if

$$\lambda_i(A)+\lambda_j(A)\neq 0$$

then

$$\lambda_i(\bar{A})+\lambda_j(\bar{A})\neq 0 \text{ for all } i \text{ and } j$$

where $\lambda_i(A)$ is the ith eigenvalue of A. Since the eigenvalues of $\bar{A}$ are the reciprocals of those of A, this completes the proof.

5. Numerical Examples

5.1 Example 1

$$\dot{x}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -23 & -72 & -61 & -61 & -39 & -11 \end{bmatrix}x+\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y=[\ 28\ 39\ 12\ 1\ 0\ ]x$$

For the fifth order SISO system of Wilson and Mishra (1979) as shown above, the impulse response errors of the reduced models are obtained in Table 1. BT and BTG give better results than RBTG for the third and first order reduced models while RBT is best among all single methods for the second order reduced model. In any case, MBT gives the best results for this example.

5.2 Example 2

For the two input two output system with eight state variables of Ozcetin et al. (1989), the impulse response errors of the reduced models are shown in Table 2. For the 7th and 6th order reduced models, RBT and RBTG give better results than BT and BTG. For the 4th, 3rd and 2nd order reduced models, however, BT and BTG give better results than RBT and RBTG. MBT gives competitive results in general, although it gives a little larger error than BT and BTG for the 4th and 2nd order reduced models.

5.3 Example 3

For the two input two output system with twelve state variables of Hung and MacFarlane

Table 1 Impulse response errors of Example 1

Order of the reduced model	BT	BTG	RBTG	RBT	MBT
3	0.00708	0.00708	0.00717	0.00717	0.00708
2	0.19537	0.19537	0.88308	0.18007	0.18006
1	0.60168	0.60168	0.88103	1.00248	0.60064

Table 2 Impulse response errors of Example 2

Order of the reduced model	BT	BTG	RBTG	RBT	MBT
7	0.03042	0.06055	0.04289	0.02874	0.02874
6	0.08300	0.05941	0.04689	0.04689	0.04689
5	0.08362	0.09182	0.07996	0.28808	0.07996
4	0.08746	0.08746	0.10819	0.29676	0.09010
3	0.21634	0.21634	0.58477	0.32390	0.21599
2	0.56369	0.56369	0.64902	0.81173	0.56404

Table 3 Impulse response errors of Example 3

Order of the reduced model	BT	BTG	RBTG	RBT	MBT	MBT2
7	0.09043	0.05362	0.05858	0.06219	0.05362	0.05858
6	0.11514	0.09438	0.06328	0.08352	0.07759	0.06328
5	0.12954	0.14714	0.08305	0.08305	0.08499	0.08305
4	0.36873	0.26578	0.15527	0.20667	0.13947	0.13821
3	0.36799	0.27316	0.25551	0.71460	0.24830	0.24763

(1982, p164), the impulse response errors of the reduced models are shown in Table 3. BTG is best among all single methods for the 7th order reduced model while RBTG is better than any other single method for the 6th, 5th, 4th and 3rd order reduced models.

For the 7th, 4th and 3rd order reduced models, MBT is better than any single method. For the 6th and 5th order reduced models, MBT is better than BT and BTG but worse than RBTG. This means that MBT does not always give better results than any single method though it usually gives good results.

A small modification can be made for MBT since the 6th order RBTG model has the smallest error among all methods. After RBTG is used to obtain the 6th order reduced model, MBT is applied to this model by reducing one order at a time by selecting one of four single methods. As shown on the final column represented as MBT2 in Table 3, this modification is better than MBT for the 6th to 3rd order reduced models.

5.4 Discussions

Generally, the errors of reduced models become larger as the number of the retained states is smaller. We can find the contradictions in our examples that the higher reduced models are not good enough to give larger errors than the lower order models, if we compare the errors of the 2nd and 1st order RBTG models of example 1, the 5th and 4th order BTG models of example 2, the 4th and 3rd BT models of example 3 and the 6th and 5th RBT models of example 3. A similar example for the BT model was suggested by Gawronski and Williams (1989). This can be explained by realizing that BT, BTG, RBT and RBTG only order the states based on Hankel singluar values and balanced gains, not on the impulse response errors.

MBT, though it uses one of four methods, usually follows the general trend that the higher order model is good enough to give smaller errors than the lower order model better than four methods. This is due to the fact that the selection of one method at each truncation is based on the impulse response error.

6. Conclusions

Two balancing reduction methods are presented. RBT truncates the states of the reciprocal balanced system corresponding to smaller Hankel singular values. MBT truncates one state at a time by selecting one of BT, BTG, RBTG and RBT, that gives the least impulse response error. The relation between the grammians of the balanced system and those of the reciprocal system has been found, thereby relieving the major computer burden of MBT.

Numerical examples show that MBT generally gives good impulse responses, although it does not always give better results than single methods and that a little modification of MBT may give better impulse response. It follows that MBT can serve as one of good methods in model reduction area because it usually gives better or competitive results compared with BT, BTG, RBT and RBTG.

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